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Hawking Radiation of Dirac Particles in Kasner-Type Spacetime

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The Hawking radiation of Dirac particles is studied in Kasner-type spacetime. Its anisotropy gives rise to particle creation.

1. INTRODUCTION

Hawking's (1974) investigations of quantum effects interpreted as the emission of a thermal spectrum of particles near a black hole event horizon has been extended by Gibbons and Hawking (1977) to the spacetime of cosmological event horizons including the de Sitter spacetime, which has attracted renewed interest as a model of the early universe. Hawking's (1974) work has been carried on by Liu Liao and Xu Dianyan (1980) into Kerr black hole spacetime and by Zhao Zheng *et al.* (1985) into Kerr–Newman black hole spacetime. Ahmed (1987) extended Hawking's (1974) work to NUT–Kerr–Newman spacetime, which includes all the asymptotically flat black hole spacetimes as well as the NUT spacetime as special cases. Xu Dianyan and Wang Huiya (1985) and Shen You-Gen (1985) extended Gibbons and Hawking's (1977) work to the Kerr–Newman–de Sitter spacetime. Recently Ahmed (1991) extended this work (Gibbons and Hawking, 1977) to the NUT–Kerr–Newman–de Sitter spacetime. Ahmed's (1987, 1991) works are interesting in that thermal radiation could also be obtained in the case of NUT–de Sitter spacetime and NUT spacetime, which have peculiar properties.

In this paper we show that thermal radiation by black holes occurs in the case of Kasner-type spacetime, an anisotropic universe.

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2. DECOUPLED EQUATION IN KASNER-TYPE SPACETIME

We consider the spacetime

$$ds^2 = r^2(d\theta^2 + \theta^2\, d\phi^2) + \Delta^{-1}\, dr^2 - \Delta\, dt^2 \tag{1}$$

where

$$\Delta = -\frac{1}{3}\Lambda r^2 - \frac{2M}{r} + \frac{e^2+g^2}{r^2}$$

Besides the cosmological constant Λ, the spacetime given by (1) contains three real parameters: the mass M, the electric charge e, and the magnetic monopole charge g.

The spacetime (1) can be transformed to the Kasner form when we put $\Lambda = e = g = 0$. Equation (1) can be written in the form

$$ds^2 = r^2(d\theta^2 + \theta^2\, d\phi^2) + \frac{r^2}{Y}\, dr^2 - \frac{Y}{r^2}\, dt^2 \tag{2}$$

where

$$Y = -\frac{\Lambda}{3} r^4 - 2Mr + e^2 + g^2$$

Kamran and McLenaghan (1984) obtained the separation of the Dirac equation in a general background. From Kamran and McLenaghan's equation, in the proper limit, we obtain the radial decoupled Dirac equation for the electron in the Kasner-type spacetime as follows:

$$\begin{aligned} &Y\frac{d^2R}{dr^2} + \left[\sqrt{Y}\frac{d}{dr}(\sqrt{Y}) - \frac{imY}{\lambda + imr}\right]\frac{dR}{dr} \\ &+ \left[K^2 Y^{-1} - \lambda^2 - m^2 r^2 + \frac{mK}{\lambda + imr} + i\sqrt{Y}\frac{d}{dr}\left(\frac{K}{\sqrt{Y}}\right)\right] = 0 \end{aligned} \tag{3}$$

$$K = r^2\omega - eQr$$

where ω is the energy of the Dirac particle, λ is the separation constant, and m and Q are the mass and the electric charge of the Dirac particle, respectively.

With the coordinate transformation

$$\frac{d}{d\hat{r}} = \frac{Y}{r^2}\frac{d}{dr} \tag{4}$$

equation (3) reduces near the horizon to the form

$$\frac{d^2R}{d\hat{r}^2}+(\omega-\omega_0)^2R=0 \tag{5}$$

where

$$\omega_0=\frac{eQ}{r_+} \tag{6}$$

and r_+, called the event horizon, is the smaller of the two positive values of r at which $Y=0$, provided the roots are real. The larger positive value of $Y=0$, denoted by r_{++}, represents the cosmological horizon.

The solution of equation (5) can easily be found to be

$$R\sim\exp[\pm i(\omega-\omega_0)]\hat{r} \tag{7}$$

Now we can write the radial wave function as

$$\psi_r=\exp[-i\omega(t\pm\hat{r}_1)] \tag{8}$$

where

$$\hat{r}_1=\frac{\omega-\omega_0}{\omega}\hat{r} \tag{9}$$

We resolve ψ_r into ingoing and outgoing waves as

$$\psi_r^{\text{in}}\sim\exp[-i\omega(t+\hat{r}_1)] \tag{10}$$

$$\psi_r^{\text{out}}\sim\exp[-i\omega(t-\hat{r}_1)] \tag{11}$$

Introducing the Eddington–Finkelstein coordinates

$$v=t+\hat{r}_1 \tag{12}$$

we obtain

$$\psi_r^{\text{in}}\sim\exp(-i\omega v) \tag{13}$$

$$\psi_r^{\text{out}}\sim\exp[-i\omega v+2i(\omega-\omega_0)\hat{r}] \tag{14}$$

Near $r=r_+$, equation (4) can be integrated to give

$$\hat{r}=\frac{1}{2\kappa_+}\ln(r-r_+) \tag{15}$$

where

$$\kappa_+ = -\frac{\Lambda}{6r_+^2}(r_+ - r_{++})(r_+ - r_-)(r_+ - r_{--}) \tag{16}$$

is the surface gravity of the event horizon of the Kasner-type spacetime. Here r_- is the inner black hole horizon and r_{--} is another cosmological horizon.

Just outside the event horizon we have

$$\psi_r^{\text{out}} \sim e^{-i\omega v}(r - r_+)^{(i/\kappa_+)(\omega - \omega_0)} \tag{17}$$

We now extend the outgoing wave outside the horizon to the region inside. Since on the event horizon the outgoing wave function is not analytic and cannot be straightforwardly extended to the region inside, it can be continued analytically to the complex plane by going around the event horizon.

Hence inside the event horizon

$$\psi_r^{\text{out}} \sim e^{-i\omega v}(r_+ - r)^{(i/\kappa_+)(\omega - \omega_0)}\, e^{(\pi/\kappa_+)(\omega - \omega_0)} \tag{18}$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{19}$$

we can generally write the outgoing wave function as

$$\phi_r^{\text{out}} = N_r \Bigg\{ y(r - r_+)\psi_r^{\text{out}}(r - r_+) + y(r_+ - r)\psi_r^{\text{out}}(r_+ - r) \exp\left[\frac{\pi}{\kappa_+}(\omega - \omega_0)\right]\Bigg\} \tag{20}$$

where ψ_r^{out} is the normalized Dirac wave function.

Expression (20) describes the splitting of ϕ_r^{out} into two components:

(a) A flow of positive-energy particles of strength N_r^2 outgoing from the event horizon.

(b) A flow of positive-energy particles in the Kasner-type background in the reverse time, since inside the event horizon, r represents the time axis due to the interchange of time and space. This can be interpreted as a flow in time of negative-energy particles ingoing toward the singularity region. This shows that a wave function near the event horizon gives rise to the creation of a Dirac particle–antiparticle pair (Deruelle and Ruffini, 1975*a*,*b*).

Obviously from the normalization condition we have

$$\langle \phi_r^{\text{out}}, \phi_r^{\text{out}} \rangle = N_r^2 \left\{ \exp\left[\frac{2\pi}{\kappa_+} (\omega - \omega_0) \right] + 1 \right\} = 1 \tag{21}$$

or

$$\begin{aligned} N_r^2 &= \left\{ \exp\left[\frac{2\pi}{\kappa_+} (\omega - \omega_0) \right] + 1 \right\}^{-1} \\ &= \left\{ \exp\left[\frac{1}{\kappa_B T_+} (\omega - \omega_0) \right] + 1 \right\}^{-1} \end{aligned} \tag{22}$$

where

$$T_+ = \frac{\kappa_+}{2\pi\kappa_B} \tag{23}$$

T_+ is the temperature of the region inside the event horizon, and κ_B is Boltzmann's constant. Equation (22) is the formula for the Hawking thermal spectrum of Dirac particles in the Kasner-type spacetime.

Following in a similar way, we have

$$T_{++} = \frac{\kappa_{++}}{2\pi\kappa_B} \tag{24}$$

where

$$\kappa_{++} = -\frac{\Lambda}{6r_{++}^2} (r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}) \tag{25}$$

is the surface gravity of the cosmological event horizon.

3. REMARKS

From this work it has appeared that an anisotropic model like Kasner-type spacetime gives rise to particle creation. The Kasner-type spacetime is due to the contraction of the Schwarzschild spacetime generalized with the cosmological constant and electric and magnetic monopole charge. This result of particle creation in the Kasner-type spacetime goes beyond the idea that in the contraction phase it is necessary that the matter should disappear. To avoid this situation, it could be said that particles do not disappear in the process of contraction but become immaterial. It will be more interesting to say that "immaterial souls" of particles are created during contraction.

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